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A THEORETICAL SOLUTION OF THE LOCKHART AND MARTINELLI FLOW MODEL FOR CALCULATING TWO-PHASE FLOW PRESSURE DROP AND HOLD-UP*

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Abstract— This paper presents a theoretical method of predicting the pressure drop and hold up in stratified and wavy two-phase flow. The theory is based on the flow model of Lockhart and Martinelli, but the theoretical solution agrees much better, compared with the generalized empirical solution developed by Lockhart and Martinelli for all flow regimes, with measurements made under these restricted conditions. A detailed calculation procedure for the theoretically developed pressure drop analysis is presented at the end of this paper.

NOMENCLATURE

C,	constant of equation (8) and (9);
D,	pipe diameter [m];
D_G,	hydraulic diameter of gas flow [m];
D_L,	hydraulic diameter of liquid flow [m];
F,	flow area [m^2];
$\tilde{F}$,	dimensionless flow area, F/R^2;
H,	average interface height [m];
H/R,	dimensionless average interface height;
L,	coordinate in flow direction [m];
M^*,	mass flow rate [kg/s];
m,	exponent of equation (8);
n,	exponent of equation (9);
$\Delta p/\Delta L$,	pressure drop per unit length [N/m^3];
$(\Delta p/\Delta L)^+$,	pressure drop per unit length for one phase at two-phase flow conditions [N/m^3];
R,	pipe radius [m];
Re,	Reynolds number;
U,	circumference [m];
$\tilde{U}$,	dimensionless circumference, $U/2R$;
$\bar{w}$,	average velocity [m/s];
X,	L and M-parameter;
α,	constant of equation (5);
β,	constant of equation (4);
$\bar{\varepsilon}$,	void fraction;
$(1 - \bar{\varepsilon})$,	hold up;
η,	dynamic viscosity [kg/ms];
ρ,	density [kg/m^3];
ϕ^2,	dimensionless two-phase flow pressure drop.

Indices

G,	gas;
L,	liquid;
2-Ph,	two-phase flow.

* Dedicated to Professor Grassmann on his 65th birthday.

1. INTRODUCTION

THE METHOD of Lockhart and Martinelli [1] (L and M method) continues to be one of the best and simplest procedures for calculating two-phase flow pressure drop and hold up. One of the biggest advantages of this procedure is that it can be used for all flow regimes. For this flexibility, however, relatively low accuracy must be accepted. Brodkey ([2], p. 471) reports a standard deviation of 40 per cent and indicates that deviations of up to 200 per cent have been found in certain flow regimes. Measurements of Bergelin and Gazley [3], Hoogendoorn [4], and Baker [5] for stratified and wavy two-phase flow confirm Brodkey's report; Bergelin and Gazley even found relative errors of more than 400 per cent in the pressure drop. The measured pressure drops are always smaller than those calculated according to L and M. Computed hold up are also in error, ranging from $\frac{1}{2}$ to $\frac{3}{4}$ of experimental values. These large discrepancies are surprising, since stratified and wavy two-phase flow are the flow geometries most closely approximated by the L and M model.

The L and M solution assumes a stratified flow model, shown in Fig. 1, and also that the

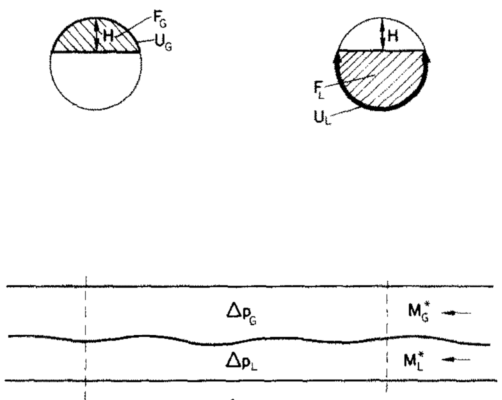

FIG. 1. Flow model.

frictional pressure drop for the liquid and gas phases is identical. The pressure drop is then computed according to standard single-phase flow computation procedures. Thus in computing the resultant L and M quantity, ϕ_G^2, the implicit assumption is made that the two-phase flow pressure drop is identical to the pressure drop of a single gas phase flowing under otherwise identical conditions. This reduced pressure drop, ϕ_G^2, is only a function of the parameter X, which, in turn, contains the transport properties and mass flow rate relationships of the two phases. Similarly, the hold up, $(1 - \bar{\varepsilon})$, is also only a function of X. L and M did not analytically solve these functions ϕ_G^2 of X and $(1 - \bar{\varepsilon})$ of X, but used experimental results to determine functional relationship.

The following work presents a theoretical solution of the equation system developed for the L and M flow model. The resulting analytic relationship between ϕ_G^2 and X as well as $(1 - \bar{\varepsilon})$ and X, are valid for the stratified and wavy two-phase flow regimes. For these flow conditions, where the L and M method is least exact, the theoretical solution agrees very well with experimental results.

2. THEORETICAL ANALYSIS

The following equations were developed for the L and M model:

$$(\Delta p/\Delta L)_{2\text{-Ph}} = (\Delta p/\Delta L)_G = (\Delta p/\Delta L)_L \quad (1)$$

$$(\Delta p/\Delta L)_G = \lambda_G \rho_G \bar{w}_G^2/(2D_G) \quad (2)$$

$$(\Delta p/\Delta L)_L = \lambda_L \rho_L \bar{w}_L^2/(2D_L) \quad (3)$$

$$F_G = \beta\pi D_G^2/4 \quad (4)$$

$$F_L = \alpha\pi D_L^2/4 \quad (5)$$

$$\bar{w}_G = M_G^*/(\rho_G F_G) \quad (6)$$

$$\bar{w}_L = M_L^*/(\rho_L F_L) \quad (7)$$

$$\lambda_G = C_G/Re_G^m = C_G/[4M_G^*/(\pi\beta D_G \eta_G)]^m \quad (8)$$

$$\lambda_L = C_L/Re_L^n = C_L/[4M_L^*/(\pi\alpha D_L \eta_L)]^n. \quad (9)$$

Substituting (4)–(9) in (2) and (3) produces:

$$(\Delta p/\Delta L)_G = (\Delta p/\Delta L)_{2\text{-Ph}} = (\Delta p/\Delta L)_G^+ \, \beta^{(m-2)} \, (D/D_G)^{(5-m)} \quad (10)$$

$$(\Delta p/\Delta L)_L = (\Delta p/\Delta L)_{2\text{-Ph}} = (\Delta p/\Delta L)_L^+ \, \alpha^{(n-2)} \, (D/D_L)^{(5-n)} \quad (11)$$

and finally

$$[(\Delta p/\Delta L)_{2\text{-Ph}}/(\Delta p/\Delta L)_G^+]^{0\cdot 5} = \phi_G = \beta^{(0\cdot 5m-1)} \, (D/D_G)^{(2\cdot 5-0\cdot 5m)} \quad (12)$$

$$[(\Delta p/\Delta L)_{2\text{-Ph}}/(\Delta p/\Delta L)_L^+]^{0\cdot 5} = \phi_L = \alpha^{(0\cdot 5-1)} \, (D/D_L)^{(2\cdot 5-0\cdot 5n)} \quad (13)$$

$$(1 - \bar{\varepsilon}) = F_L/(\pi D^2/4) = \alpha(D_L/D)^2. \quad (14)$$

Where $(\Delta p/\Delta L)_{2\text{-Ph}}$ indicates two-phase flow pressure drop, and the pressure drop for pure gas and liquid flow under otherwise identical conditions is indicated, respectively, by $(\Delta p/\Delta L)_G^+$ and $(\Delta p/\Delta L)_L^+$. D is the pipe diameter, and D_G and D_L are the hydraulic diameters for, respectively, the gas and liquid phases, α and β are correction factors which account for discrepancies between D_G and D_L and the actual flow geometry.

As can be seen, the quantities ϕ_G, ϕ_L, and $(1 - \bar{\varepsilon})$ are functions only of D_G/D, D_L/D, α, and β. L and M postulate, in accordance with measurements, that all of these variables are functions of the single variable X, where X is defined as:

$$X = [(\Delta p/\Delta L)_L^+/(\Delta p/\Delta L)_G^+]^{0\cdot 5} = \left[\frac{M_L^{*(1-0\cdot 5n)}}{M_G^{*(1-0\cdot 5m)}}\right]\left[\frac{\eta_L^{0\cdot 5n}}{\eta_G^{0\cdot 5m}}\right]\left[\frac{\rho_L}{\rho_G}\right]^{-0\cdot 5}\left[\frac{C_L}{C_G}\right]^{0\cdot 5}. \quad (15)$$

But, according to (12) and (13), X can also be defined as

$$X = \phi_G/\phi_L. \quad (16)$$

The relationships determined by L and M between ϕ_G and X, and $(1 - \bar{\varepsilon})$ and X are shown in Figs. 2 and 3. These curves are valid for turbulent flow in both phases: $C_L = C_G$ and $n = m = 0\cdot 25$. The relationship between ϕ_L and X is not presented, since ϕ_L and ϕ_G are related through (16). A theoretical solution of the above problem is possible through the solution of equations (12)–(14). In order to do this, the hydraulic diameter for both phases must be determined.

It can be shown:

$$D_G = 4F_G/U_G \quad (17)$$

$$D_L = 4F_L/U_L. \quad (18)$$

F_G and F_L are the respective cross-sectional flow areas, and U_G and U_L the respective circumferences of the gas and liquid phases. Assumptions are necessary in order to determine U_G and U_L. It is reasonable to assume that friction

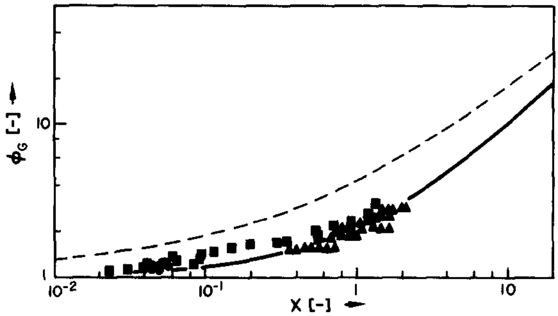

FIG. 2. Two-phase pressure drop for turbulent-turbulent flow; key, ---- general correlation of Lockhart and Martinelli, —— present theory for wavy and stratified flow. ▲ Bergelin and Gazley [3], ■ Hoogendoorn [4], ● Baker [5]. The measurements are in the regions of wavy and stratified flow.

between the gas and the pipe wall as well as liquid surface is equal, since two-phase flow gas velocities are very much higher than that of the liquid. Also, frictional pressure drop in the liquid can be calculated by open-channel flow computation procedures. Accordingly, U_G and U_L are shown in Fig. 1. Denoting the mean distance between the liquid–gas interface and the top of the pipe by H, and the pipe radius by R, the following geometric relationships can be developed:

$$F_G = R^2\{\arccos(1 - H/R) - (1 - H/R) \times \sqrt{[2H/R - (H/R)^2]}\} = R^2\tilde{F}_G \quad (19)$$

$$F_L = R^2\{\pi - \arccos(1 - H/R) + (1 - H/R) \times \sqrt{[2H/R - (H/R)^2]}\} = R^2 F_L \quad (20)$$

$$U_G = 2R\{\arccos(1 - H/R) + \sqrt{[2H/R - (H/R)^2]}\} = 2R\tilde{U}_G \quad (21)$$

$$U_L = 2R[\pi - \arccos(1 - H/R)] = 2R\tilde{U}_L. \quad (22)$$

Substituting equations (19)–(22) into (17) and (18) produces the following relationships for D_G and D_L:

$$D_G = 2R\tilde{F}_G\tilde{U}_G^{-1} \quad (17')$$

$$D_L = 2R\tilde{F}_L\tilde{U}_L^{-1}. \quad (18')$$

Equations (4) and (5) can now be written, through use of equations (17′) and (18′) as follows:

$$\beta = \tilde{F}_G^{-1}\tilde{U}_G^2\pi^{-1} \quad (4')$$

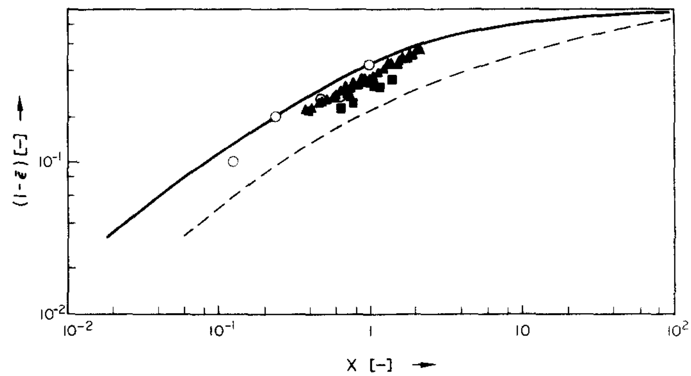

FIG. 3. Hold up for turbulent–turbulent two-phase flow: key, ---- general correlation of Lockhart and Martinelli, —— present theory for wavy and stratified flow, ▲ Bergelin and Gazley [3], ■ Hoogendoorn [4], ○ Govier and Omer [6]. The measurements are in the regions of wavy and stratified flow.

$$\alpha = \tilde{F}_L^{-1}\tilde{U}_L^2\pi^{-1}. \quad (5')$$

The functional relationship between D_G, D_L, β and α and the geometric variable H/R is shown by equations (17′), (18′), (4′) and (5′). All of the variables are now seen to be a function of H/R since ϕ_G, ϕ_L, $(1-\bar{\varepsilon})$ and X are in turn related to D_G, D_L, β and α through equations (12)–(14) and (16). Thus:

$$\phi_G = \pi^{(1-0\cdot5m)}\tilde{U}_G^{(0\cdot5m+0\cdot5)}\tilde{F}_G^{-1\cdot5} \quad (12')$$

$$\phi_L = \pi^{(1-0\cdot5n)}\tilde{U}_L^{(0\cdot5n+0\cdot5)}\tilde{F}_L^{-1\cdot5} \quad (13')$$

$$(1-\bar{\varepsilon}) = \tilde{F}_L/\pi \quad (14')$$

$$X = \phi_G/\phi_L = [\pi^{(0\cdot5n-0\cdot5m)}]\,[\tilde{U}_G^{(0\cdot5m+0\cdot5)}] \times [\tilde{U}_L^{(0\cdot5-0\cdot5n)}]\,[\tilde{F}_G^{-1\cdot5}]\,[\tilde{F}_L^{1\cdot5}]. \quad (16')$$

Equations (12′), (14′) and (16′) analytically prove that ϕ_G and $(1-\bar{\varepsilon})$ are a function of only the variable X, as postulated by L and M. It is now possible to use equation (16′) to eliminate H/R in equations (12′) and (14′). The resultant theoretical relationship between ϕ_G and X is presented in Fig. 2. The theoretically predicted hold up, $(1-\bar{\varepsilon})$, is shown in Fig. 3. These curves too are valid for turbulent flow in the gas and liquid; for these curves $n = m = \frac{1}{4}$ and $C_G = C_L$. This theoretical solution is only valid for the case of turbulent flow, since the hydraulic diameter has been used. Two-phase laminar flow, however, is very rare.

The turbulent flow condition requires that $Re > 2000$ and can be written as follows:

$$M_G^*/(10^3\eta_G R) > \tilde{U}_G \quad (23)$$

$$M_L^*/(10^3\eta_L R) > \tilde{U}_L. \quad (24)$$

The relationship between $\tilde{U}_G$, $\tilde{U}_L$ and X is shown in Fig. 4. With the help of this diagram it is possible to determine whether the flow is laminar or turbulent.

3. COMPARISON BETWEEN THEORY AND MEASUREMENT

Figure 2 presents a comparison between pressure drops predicted by analysis developed in this paper, the L and M method, and measurements made in the stratified and wavy two-phase flow region. The data presented here are from the previously mentioned works of Bergelin and Gazley [3], Hoogendoorn [4], and Baker [5]. Pipe diameters of 52·5 mm, 140 mm and 197 mm were employed and the flow systems consisted of water/air, oil/air, and oil/natural gas.

The previously mentioned large discrepancy between the L and M curve and measurements can be clearly seen. It should be noted that the pressure drop is proportional to ϕ_G^2. Thus the new theoretical curve represents a significant

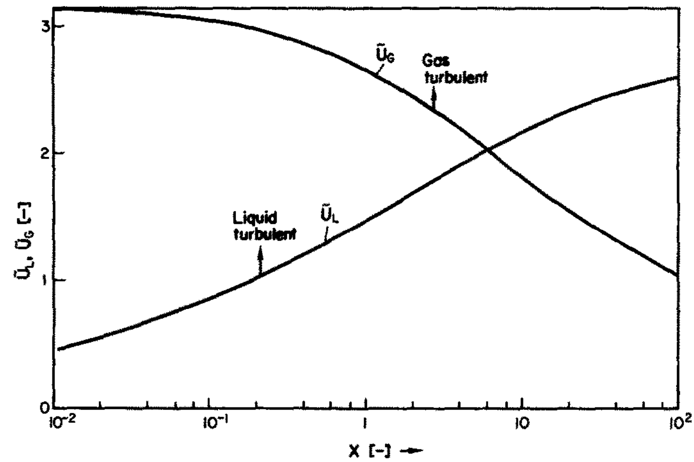

FIG. 4. Regimes of turbulent flow.

improvement on the L and M curve. For values of X between 0·3 and 2, the data points are relatively evenly distributed about the theoretical curve. For low values of X, i.e. large mass flow rates of gas, the theoretical curve is somewhat low. Some deviation in this region is, however, to be expected for several reasons. First, wave development due to high gas velocities is fairly intense in this region, requiring large energy transfer from the gas to the wave system. This energy transfer produces a gas phase pressure drop which was not considered in the theory.

Additionally, the liquid phase receives a thrust from the gas phase which was also not considered in the theory since the liquid was assumed to be flowing in an open channel. Thus the computed liquid phase pressure drop is too large. This could only be reduced by increasing the liquid flow area which, in turn, would increase the gas phase pressure drop. It should also be considered that some of these data points were probably measured in flow systems with breaking waves, since many authors so not distinguish between this and wave flow. In the case of breaking waves, liquid droplets would be accelerated by the flowing gas, representing an additional gas phase energy loss.

The difference between the ϕ_G curves computed by L and M and in this analysis are quite apparent, despite the fact that this analysis is based on the L and M model. This difference can be explained as follows: in one case an emperical relationship was developed, where as in the other case an analytical relationship, based on very realistic assumptions, was developed. It should be noted here that the theory is limited to a stratified flow. Only under this condition can it be theoretically shown that X is the single independent variable. Application of the L and M method is not limited to only one flow regime, since the development of their relationship between ϕ_G and X was based on data from all two-phase flow geometries. It can be shown, however, that for certain flow regimes ϕ_G is not dependent only on X.

Measurements of Hoogendoorn ([4], Fig. 8) show that, although X remains constant, the pressure drop, and thus ϕ_G^2, increase by a factor of 2·8 with the transition from wavy to slug flow. This change in pressure drop is to be expected since, at the flow regime transition, liquid must be rapidly accelerated by the flowing gas. Good agreement is found between the L and M curve and measured data in the slug flow regime, which indicates that their emperical relationship was strongly influenced by measurements outside the stratified flow regime. Thus the largest deviation between the L and M curve and measurements is found in the flow regime where their model is most realistic.

A similar comparison is shown for hold up in Fig. 2. The measured data lies between the two curves, as was the case for pressure drop, and the theoretical curve is again seen to be the much better approximation. The variation between theory and measurements is very probably related to the fact that, as Bergelin and Gazley have shown, the hold up is not constant along the length of the pipe, but decreases as the free end of the pipe is approached. At the free or open end of the pipe, the hold up reaches a minimum since the liquid level there approaches the minimum or critical level. The theoretical model would be valid for an infinitely long pipe and, in practice, is best approximated near the pipe entrance where the liquid level is relatively constant. The empirical data [3] were measured near the pipe middle.

As was shown for pressure drop, the hold up experiences step variations at the transitions from stratified to plug as well as to slug flow. The reason for this is that a portion of the liquid is suddenly accelerated to the gas velocity. Since L and M used the same data for their hold up analysis as was used for pressure drop, the error in their curve can again be explained by the fact that the majority of their data was outside of the range in which their theory was valid.

4. APPLICATION OF THE THEORY

This new theoretical solution of the L and M flow model can be used to predict pressure drop and void fraction for stratified and wavy two-phase flow, where the L and M method is least exact. The range of validity for this theory is bounded on one extreme by the turbulent flow requirement (Fig. 4), and on the other extreme by the onset of slug, plug, or breaking wave flow. This latter extreme can be determined by the flow regime diagrams of Schicht [7], Hoogendoorn [4] or Baker [5].

Application of the theory presented in this paper follows exactly the same procedure as for the L and M method. The variable X is comuted according to equation (15), where $n = m = \frac{1}{4}$ and $C_G = C_L$. ϕ_G and $(1 - \bar{\varepsilon})$ may then be read from Figs. 2 and 3. According to the defining equation (12)

$$(\Delta p/\Delta L)_{2\text{-Ph}} = \phi_G^2(\Delta p/\Delta L)_G^+ \qquad (12)$$

$$(\Delta p/\Delta L)_G^+ = 8\lambda_G M_G^{*2}/(\pi^2 D^5 \rho_G). \qquad (25)$$

Thus defining the two-phase flow pressure drop.

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RESOLUTION THEORIQUE DU MODELE D'ECOULEMENT DE LOCKHART ÈT MARTINELLI POUR LE CALCUL DE LA CHUTE DE PRESSION ET DE LA CAPTATION DANS UN ECOULEMENT BIPHASIQUE

Résumé— Ce mémoire présente une méthode théorique de calcul de la chute de pression et de la captation dans un écoulement biphasique stratifié et onduleux. La théorie est basée sur le modèle de Lockhart et Martinelli mais la solution théorique s'accorde avec les mesures, faites dans le cadre des conditions considérées, mieux que la solution empirique généralisée développée par Lockhart et Martinelli pour tous les regimes d'écoulement. Une procédure détaillée de calcul par voie théorique de la chute de pression est présentée à la fin de ce papier.

EINE THEORETISCHE LÖSUNG DES LOCKHART UND STRÖMUNGSMODELLS ZUR BERECHNUNG VON DRUCKABFALL UND FLÜSSIGKEITSINHALT BEI ZWEIPHASENSTRÖMUNGEN

Zusammenfassung—Es wird eine Methode zur Vorausberechnung des Druckabfalls und des Flüssigkeitsinhalts in geschichteter und welliger Zweiphasenströmung angegeben. Die Theorie beruht auf dem Strömungsmodell von Lockhart und Martinelli, ihre Ergebnisse stimmen aber viel besser mit Messungen unter diesen eingeschränkten Bedingungen überein, verglichen mit der verallgemeinerten empirischen Lösung von Lockhart und Martinelli für alle Strömungszustände. Am Ende der Arbeit wird ein detailliertes Rechenverfahren für den Druckabfall mitgetallt.

ТЕОРЕТИЧЕСКИЙ МЕТОД РАСЧЕТА ПЕРЕПАДА ДАВЛЕНИЯ И ЕГО ПОДДЕРЖАНИЯ В ДВУХФАЗНОМ ПОТОКЕ МОДЕЛИ ЛОКХАРДА И МАРТИНЕЛЛИ

Аннотация—В работе предложен теоретический метод расчета перепада давления и его поддержания в стратифицированном и волнообразном двухфазных потоках. Теория построена на модели течения Локхарта и Мартинелли, однако теоретическое решение согласовывается с измерениями, проведенными в ограниченных условиях, намного лучше, чем обобщенное эмпирическое решение, полученное Локхартом и Мартинелли для всех режимов течения. В конце доклада приводится подробная методика расчета перепада давления по предложенной теории.